JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **220,** 603–612 (1998)
ARTICLE NO. AY975855

# A Representation of Projection Lattices and Their States in Euclidean Space

Bob Coecke*

*FUND-DWIS, Free University of Brussels, Pleinlaan 2, B-1050, Brussels, Belgium*



Received March 19, 1996

We propose a representation $r: \mathscr{L} \cup \Omega \to \mathbb{R}^\nu$, where $\mathscr{L}$ is the collection of closed subspaces of an $n$-dimensional real, complex, or quaternionic Hilbert space $\mathscr{H}$, or equivalently, the projection lattice of this Hilbert space, where $\Omega$ is the set of all states $\omega: \mathscr{L} \to [0,1]$. The value that $\omega \in \Omega$ takes in $a \in \mathscr{L}$ is given by the scalar product of the representative points $(r(a)$ and $r(\omega))$. The representation $r(a \vee b)$ of the join of two orthogonal elements $a, b \in \mathscr{L}$ is equal to $r(a) + r(b)$. The convex closure of the representation of $\Sigma$, the set of atoms of $\mathscr{L}$, is equal to the representation of $\Omega$. © 1998 Academic Press

## 1. INTRODUCTION

A purely structural description of quantum mechanics has been pushed forward by Birkhoff and von Neumann [1]. This approach is now known as the logico-algebraic approach to quantum mechanics [2–4]. The primal concepts in it are the existence of an *orthocomplementation* and of a *non-distributive algebraic structure* on the set $\mathscr{L}$ of all closed subspaces of a Hilbert space, or equivalently, the projection lattice of this Hilbert space (the collection of orthogonal projectors on these subspaces). Other structural approaches have meanwhile been developed, and one of the most important of them is the convex approach, pushed forward by Segal [5], and developed in its details around 1970 [6–9]. The primal concept in this

* E-mail: bocoecke@vub.ac.be.



0022-247X/98 $25.00

approach is the *convex structure* of the space $\Omega$ of all states $\omega : \mathscr{L} \to [0, 1]$.[1] Both approaches have undergone a separate development, because they deal with different mathematical representation spaces.[2]

In this paper we introduce a representation $r$ of $\mathscr{L} \cup \Omega$ in $\mathbb{R}^\nu$, i.e., for every finite dimensional real, complex, or quaternionic Hilbert space[3] $\mathscr{H}$ we represent every subspace in $\mathscr{L}$ and every state in $\Omega$ as a point of $\mathbb{R}^\nu$ (we remark that an atom $p \in \mathscr{L}$ and the state that takes the value 1 in $p$ are represented by the same point of $\mathbb{R}^\nu$). For all $\omega \in \Omega$, the value that $\omega$ takes in $a \in \mathscr{L}$ will be given by the scalar product $r(\omega) \cdot r(a)$. This condition implies the following minimal dimensions of the representation space $\mathbb{R}^\nu$ (we consider an $n$-dimensional Hilbert space $\mathscr{H}$): if $\mathscr{H}$ is over $\mathbb{R}$ then $\nu \geq n(n+1)/2$, if $\mathscr{H}$ is over $\mathbb{C}$ then $\nu \geq n^2$, and if $\mathscr{H}$ is over $\mathbb{H}$ then $\nu \geq n(2n-1)$.

Through this representation, we join the logico-algebraical aspect and the convex aspect of the Hilbert space formalism within one representation space. Other attempts to represent the complex quantum formalism in a real space have already been made in the past [12]. Still, these approaches had completely different goals than the one in this paper, i.e., a representation of two different structural aspects of the Hilbert space formalism within one representation space. Moreover, the representation that is introduced in this paper has some very remarkable extra features that make it interesting from a purely mathematical point of view. We also remark that some results presented in this paper have already been published in previous papers [14, 15]. These papers deal with interpretative aspects of quantum mechanics,[4] and thus, they take a completely different point of view on the conceptual level and on the formal level.

[1] One should be aware of the fact that according to some important authors, this concept of state which can be found in books which study quantum-like structures differs from what can be considered as the physical states of an entity. They consider these states that take the value one in an atom of the lattice (called pure states) as the physical states (see [3]). In order to be in accordance with most of the mathematical literature on this subject, the present author chooses for the purpose of this paper to follow the definition of [4].

[2] These approaches also start with a different conception concerning the fundamental meaning of the mathematical objects encountered in quantum theory, but since it is the aim of this paper to focus on purely structural aspects of the Hilbert Space formalism, we won't give any attention to these more philosophical aspects.

[3] For more details on the quaternionic Hilbert space formalism we refer to [10, 11].

[4] Some results in this paper have applications to the hidden measurement approach to quantum mechanics, introduced in [13]. The implementation of some results presented in this paper within this approach can be found in [14, 15].

## 2. REPRESENTATION OF $\mathscr{L} \cup \Omega$ IN $\mathbb{R}^{\nu}$

If $\mathscr{L}$ consists of the closed subspaces of a finite dimensional real, complex, or quaternionic Hilbert space, it turns out to be a complete atomic orthomodular lattice with the covering property[5] (see Piron's representation theorem which can be found in [16, 17]). Let 1 be the supremum of $\mathscr{L}$, let 0 be the infimum of $\mathscr{L}$, and let $\Sigma$ be the set of atoms of $\mathscr{L}$, i.e., $\Sigma = \{p \in \mathscr{L} \mid p \neq 0, \ \forall a \in \mathscr{L}: a < p \Rightarrow a = 0\}$. For all $p \in \Sigma$, we choose a representative unit vector $\psi_p = \{\psi_p^1, \ldots, \psi_p^n\} \in p$. The collection $\Omega$ of states consists of all $\omega: \mathscr{L} \to [0,1]$ which fulfill $\omega(1)$ and $\forall a, b \in \mathscr{L}$ with $a \perp b = 0$, $\omega(a \vee b) = \omega(a) + \omega(b)$, where $a \perp b = 0 \Leftrightarrow a \leq b'$ with $b'$ the orthocomplement of $b$. We introduce a representation of $\Sigma$ in the following way

- If $\mathscr{H}$ is a real Hilbert space, define $r: \Sigma \to \mathbb{R}^{n(n+1)/2}: p \mapsto (x_1, \ldots, x_{n(n+1)/2})$ where

$$2 \leq i \leq n, 1 \leq j < i: x_{i(i-1)/2-j+1} = \sqrt{2}\,\psi_p^i \psi_p^j \tag{1}$$

$$1 \leq i \leq n: x_{n(n-1)/2+i} = \left(\psi_p^i\right)^2. \tag{2}$$

- If $\mathscr{H}$ is a complete Hilbert space, define $r: \Sigma \to \mathbb{R}^{n^2}: p \mapsto (x_1, \ldots, x_{n^2})$ where

$$2 \leq i \leq n, 1 \leq j < i: x_{i(i-1)-2j+1} = \sqrt{2}\ \mathrm{Re}\left(\bar{\psi}_p^i \psi_p^j\right) \tag{3}$$

$$2 \leq i \leq n, 1 \leq j < i: x_{i(i-1)-2j+2} = \sqrt{2}\ \mathrm{Im}\left(\bar{\psi}_p^i \psi_p^j\right) \tag{4}$$

$$1 \leq i \leq n: x_{n(n-1)+i} = \left(\psi_p^i\right)^2. \tag{5}$$

- If $\mathscr{H}$ Is a quaternionic Hilbert space, define $r: \Sigma \to \mathbb{R}^{n(2n-1)}: p \mapsto (x_1, \ldots, x_{n(2n-1)})$ where

$$2 \leq i \leq n, 1 \leq j < i: x_{2i(i-1)-4j+1} = \sqrt{2}\ \mathrm{Re}\left(\bar{\psi}_p^i \psi_p^j\right) \tag{6}$$

$$2 \leq i \leq n, 1 \leq j < i: x_{2i(i-1)-4j+2} = \sqrt{2}\ \mathrm{Im}_1\left(\bar{\psi}_p^i \psi_p^j\right) \tag{7}$$

$$2 \leq i \leq n, 1 \leq j < i: x_{2i(i-1)-4j+3} = \sqrt{2}\ \mathrm{Im}_2\left(\bar{\psi}_p^i \psi_p^j\right) \tag{8}$$

$$2 \leq i \leq n, 1 \leq j < i: x_{2i(i-1)-4j+4} = \sqrt{2}\ \mathrm{Im}_3\left(\bar{\psi}_p^i \psi_p^j\right) \tag{9}$$

$$1 \leq i \leq n: x_{2n(n-1)+i} = \left(\psi_p^i\right)^2. \tag{10}$$

[5] Every irreducible complete atomic orthomodular lattice with finite chains that satisfies the covering law is isomorphic with the projection lattice of a Hilbert space over a division ring. To get $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the division ring must be connected and locally compact.

One easily verifies that these representations are well defined and one to one (for the case of a complex Hilbert space, this is done in [14]). Let $\nu = n(n+1)/2$, $\nu = n^2$, or $\nu = n(2n-1)$, depending on the division ring of $\mathscr{H}$, and denote $\{r(p) | p \in \Sigma\}$ by $r(\Sigma)$. We have the following proposition which states that $r(\Sigma)$ spans $\mathbb{R}^\nu$ as a vector space.

PROPOSITION 1. *$r(\Sigma)$ spans $\mathbb{R}^\nu$.*

*Proof.* Let $\nu = n(2n-1)$. Substitute $\psi_p$ in relations (6), (7), (8), (9), and (10) by following the vectors of $\mathbb{H}^n$:

(i) $(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)$

(ii) $(1,1,0,\dots,0), (1,0,1,0,\dots,0), \dots, (1,0,\dots,0,1)$; $(0,1,1,0,\dots,0), (0,1,0,1,0,\dots,0), \dots, (0,1,0,\dots,0,1); \dots; (0,0,\dots,0,1,1)$

(iii) $(1,i,0,\dots,0), (1,0,i,0,\dots,0), \dots, (1,0,\dots,0,i)$; $(0,1,i,0,\dots,0), (0,1,0,i,0,\dots,0), \dots, (0,1,0,\dots,0,i); \dots; (0,0,\dots,0,1,i)$

(iv) $(1,j,0,\dots,0), (1,0,j,0,\dots,0), \dots, (1,0,\dots,0,j)$; $(0,1,j,0,\dots,0), (0,1,0,j,0,\dots,0), \dots, (0,1,0,\dots,0,j); \dots; (0,0,\dots,0,1,j)$

(v) $(1,k,0,\dots,0), (1,0,k,0,\dots,0), \dots, (1,0,\dots,0,k)$; $(0,1,k,0,\dots,0), (0,1,0,k,0,\dots,0), \dots, (0,1,0,\dots,0,k); \dots; (0,0,\dots,0,1,k)$.

One easily verifies that we obtain $n(2n-1)$ linear independent vectors in $\mathbb{R}^{n(2n-1)}$. If $\nu = n^2$, we consider the vectors (i), (ii), and (iii) and if $\nu = n(n+1)/2$, we consider the vectors (i) and (ii). ∎

Thus, the dimension represented by $\nu$ is the smallest possible to enable the representation (with respect to the Euclidean structure of $\mathbb{R}^\nu$). Moreover, we are also able to generalize Proposition 1 to the subsets of $r(\Sigma)$ that correspond with subspaces of $\mathscr{H}$. Before we do this we prove some lemmas.

LEMMA 1. *For all $h, h' \in \mathbb{H}$:*

(i) $\mathrm{Re}(hh') = \mathrm{Re}(h'h)$

(ii) $\mathrm{Re}(\bar{h}h') = \mathrm{Re}(h)\mathrm{Re}(h') + \sum_{i=1}^{i=3} \mathrm{Im}_i(h)\mathrm{Im}_i(h')$.

*Proof.* Let $I = \{i, j, k\}$, $h = x + \sum_{\alpha \in I} \alpha x_\alpha$, and $h' = x' + \sum_{\alpha \in I} \alpha x'_\alpha$, with $x, x', x_\alpha, x'_\alpha$ in $\mathbb{R}$.

$$\mathrm{Re}(hh') = \mathrm{Re}\left( xx' + x \sum_{\alpha \in I} \alpha x'_\alpha + x' \sum_{\alpha \in I} \alpha x_\alpha + \sum_{\alpha \in I} \sum_{\beta \in I} \alpha \beta x_\alpha x'_\beta \right)$$
$$= xx' - \sum_{\alpha \in I} x_\alpha x'_\alpha = x'x - \sum_{\alpha \in I} x'_\alpha x_\alpha = \mathrm{Re}(h'h).$$

$\mathrm{Re}(\bar{h}h') = xx' + \sum_{\alpha \in I} x_\alpha x'_\alpha = \mathrm{Re}(h)\mathrm{Re}(h') + \sum_{i=1}^{i=3} \mathrm{Im}_i(h)\mathrm{Im}_i(h')$. ∎

LEMMA 2. *For all* $p, q \in \Sigma$, $|\langle \psi_p | \psi_q \rangle| = \sqrt{r(p) \cdot r(q)}$.

*Proof.* We consider a quaternionic Hilbert space (the proof of this case implies the lemma for complex and real Hilbert spaces). Since $\psi_p^i \overline{\psi}_q^i \psi_q^j \overline{\psi}_p^j = \overline{\overline{\psi_q^j \overline{\psi}_p^j}\;\overline{\psi_p^i \overline{\psi}_q^i}} = \overline{\psi_p^j \overline{\psi}_q^i \psi_q^i \overline{\psi}_p^i}$ and by applying Lemma 1 we have

$$
\begin{aligned}
|\langle \psi_p | \psi_q \rangle|^2 &= \sum_{i,j} \psi_p^i \overline{\psi}_q^i \overline{\psi_p^j \overline{\psi}_q^j} \\
&= \sum_i \psi_p^i \overline{\psi}_q^i \psi_q^i \overline{\psi}_p^i + \sum_{j<i} \left( \psi_p^j \overline{\psi}_q^j \psi_q^i \overline{\psi}_p^i + \psi_p^i \overline{\psi}_q^i \psi_q^j \overline{\psi}_p^j \right) \\
&= \sum_i \overline{\psi}_p^i \psi_p^i \overline{\psi}_q^i \psi_q^i + 2 \sum_{j<i} \mathrm{Re}\left( \psi_p^j \overline{\psi}_q^j \psi_q^i \overline{\psi}_p^i \right) \\
&= \sum_i \overline{\psi}_p^i \psi_p^i \overline{\psi}_q^i \psi_q^i + 2 \sum_{j<i} \mathrm{Re}\left( \overline{\psi}_p^i \psi_p^j \overline{\psi}_q^j \psi_q^i \right) \\
&= \sum_i \overline{\psi}_p^i \psi_p^i \overline{\psi}_q^i \psi_q^i + 2 \sum_{j<i} \Bigg[ \mathrm{Re}\left( \overline{\psi}_p^i \psi_p^j \right) \mathrm{Re}\left( \overline{\psi}_q^i \psi_q^j \right) \\
&\qquad\qquad + \sum_{k=1,2,3} \mathrm{Im}_k\left( \overline{\psi}_p^i \psi_p^j \right) \mathrm{Im}\left( \overline{\psi}_q^i \psi_q^j \right) \Bigg] \\
&= r(p) \cdot r(q). \quad \blacksquare
\end{aligned}
$$

This second lemma tells us that the square of the Hilbert in-product (i.e., the transition probability in quantum mechanics) is given in the representation by the Euclidean scalar product.

Let $a \in \mathscr{L}$ and let $\Sigma_a = \{p | p < a\}$. Denote the dimension of $a$ by $\dim(a)$, and let $\nu_k = k(k+1)/2$ if $\nu = n(n+1)/2$, $\nu_k = k^2$, if $\nu = n^2$, and $\nu_k = k(2k-1)$ if $\nu = n(2n-1)$. Now we generalize Proposition 1 to certain subsets of $r(\Sigma)$.

PROPOSITION 2. $r(\Sigma_a)$ *spans a* $\nu_{dim(a)}$*-dimensional subspace of* $\mathbb{R}^\nu$,

*Proof.* If $a$ is spanned by a subset of the vectors (i) (see Proposition 1), it suffices to consider the vectors (i), (ii) (and (iii), (iv), (v), depending on $\mathscr{H}$). If $a$ is not spanned by a subset of the vectors (i), there exists a unitary transformation $U$ that transforms a subset of the vectors (i) into an orthonormal set than spans $a$. Define $O : r(\Sigma) \to r(\Sigma)$ such that $O(r(p)) = r(p')$ if $U(\psi_p) = \psi_{p'}$. If $\psi_{p'} = U(\psi_p)$ and $\psi_{q'} = U(\psi_q)$,

$$
\begin{aligned}
r(p) \cdot r(q) &= |\langle \psi_p | \psi_q \rangle|^2 = |\langle U(\psi_p) | U(\psi_q) \rangle|^2 = |\langle \psi_{p'} | \psi_{q'} \rangle|^2 \\
&= r(p') \cdot r(q') = O(r(p)) \cdot O(r(q))
\end{aligned}
$$

(see Lemma 2). Since $r(\Sigma)$ spans $\mathbb{R}^{\nu}$ (see Proposition 1) and $O$ preserves the angles in $r(\Sigma)$, it suffices to consider the vectors (i), (ii) (and (iii), (iv), (v)), after a transformation by $O$. ∎

Let $\{\psi_{p_1}, \ldots, \psi_{p_k}\}$ be an orthonormal set of vectors that span the subspace $a$ where $\{p_1, \ldots, p_k\}$ is the corresponding set of atoms in $\Sigma_a$. We extend the domain of $r$ to $\mathscr{L}$ in the following way:

- Define $r : \mathscr{L} \to \mathbb{R}^{\nu} : a \mapsto \sum_{i=1}^{i=k} r(p_i)$.

We have to prove that $\forall a \in \mathscr{L}, r(a)$ does not depend on the choice of the base $\{\psi_{p_1}, \ldots, \psi_{p_k}\}$. ∎

LEMMA 3. *If $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ correspond with two different orthonormal bases of $a$ (in the sense explained above), then $\sum_{i=1}^{i=k} r(p_i) = \sum_{i=1}^{i=k} r(q_i)$.*

*Proof.* By Proposition 2 we know that the subspace spanned by $r(\Sigma_a)$ is $\nu_{dim(a)}$-dimensional. Let $\{x_1, \ldots, x_{dim(a)}\} \subset r(\Sigma_a)$ be a base of this subspace. Since $\sum_{i=1}^{i=k} r(p_i)$ and $\sum_{i=1}^{i=k} r(q_i)$ belong to this subspace and $\forall p \in \Sigma_a$, $r(p)\sum_{i=1}^{i=k} r(p_i) = \sum_{i=1}^{i=k} r(p) r(p_i) = \sum_{i=1}^{i=k} |\langle \psi_p | \psi_{p_i} \rangle|^2 = 1$, we have $\forall j$, $x_j \sum_{i=1}^{i=k} r(p_i) = 1 = x_j \sum_{i=1}^{i=k} r(q_i)$, i.e., $\sum_{i=1}^{i=k} r(p_i)$ and $\sum_{i=1}^{i=k} r(q_i)$ have the same coordinates in the base $\{x_1, \ldots, x_{dim(a)}\}$, i.e., they are equal. ∎

This also means that the complete Hilbert space $\mathscr{H}$ is represented by one unique point in the Euclidean space, namely the sum of the representations of an orthonormal base of $\mathscr{H}$.

Suppose the dimension of $\mathscr{H}$ is at least three.[6] We extend the domain of $r$ to $\mathscr{L} \cup \Omega$ in the following way:

- Let $\omega \in \Omega$. As a consequence of the Gleason theorem [18],[7] there always exists an orthonormal base $\{\psi_{p_1^{\omega}}, \ldots, \psi_{p_n^{\omega}}\}$ of $\mathscr{H}$, with $\{p_1^{\omega}, \ldots, p_n^{\omega}\}$ the corresponding set in $\Sigma$, where the density matrix that corresponds with $\omega$ is diagonal. In this base, $\{\omega(p_1^{\omega}), \ldots, \omega(p_n^{\omega})\}$ defines $\omega$ completely. Define $r : \mathscr{L} \cup \Omega \to \mathbb{R}^{\nu}$ such that $\forall \omega \in \Omega, r(\omega) = \sum_{i=1}^{i=n} \omega(p_i^{\omega}) r(p_i^{\omega})$.

[6] In the case that $\mathscr{H}$ is two dimensional, there exist maps $\omega : \mathscr{L} \to [0, 1]$ in $\Omega$ that do not correspond to a density matrix, due to the fact that the two dimensional lattice hasn't got a sufficiently rich structure to impose a Gleason-like theorem. However, if we would limit ourselves to those $\omega \in \Omega$ which do have a representation as a density matrix, the representation introduced in this paper would also work for the two dimensional case.

[7] We remark that Gleason's theorem is also valid for quaternionic Hilbert spaces. For a proof we refer to [17].

## 3. THE STRUCTURE OF $r(\mathscr{L} \cup \Omega)$

We have the following three theorems that express some logico-algebraical aspects of $\mathscr{H}$ within the representation space (for all these results we suppose that the dimension of $\mathscr{H}$ is at least three).

THEOREM 1a. *For all $a, b \in \mathscr{L}$, $a \perp b \Leftrightarrow r(a) \cdot r(b) = 0$.*

*Proof.* Let $\{p_1, \ldots, p_{dim(a)}\}$ correspond with a base of $a$ and let $\{q_1, \ldots, q_{dim(b)}\}$ correspond with a base of $b$. Since $r(a) \cdot r(b) = \sum_{i=1}^{i=dim(a)} r(p_i) \cdot \sum_{i=1}^{i=dim(b)} r(q_i) = \sum_{i=1}^{i=dim(a)} \sum_{j=1}^{j=dim(b)} r(p_i) \cdot r(q_j)$ and $r(p_i) \cdot r(q_j) \geq 0$ (see Lemma 2) we have $r(a) \cdot r(b) = 0 \Leftrightarrow \forall i, \forall j: r(p_i) \cdot r(q_j) = 0 \Leftrightarrow \forall i,\ \forall j: |\langle \psi_{p_i} | \psi_j \rangle| = 0 \Leftrightarrow \forall i,\ \forall j: p_i \perp p_j \Leftrightarrow a \perp b$, again as a consequence of Lemma 2. ∎

Thus, the orthogonality of the rays of $\mathscr{H}$ is preserved. Moreover, the orthogonality of the subspaces of $\mathscr{H}$ (in the Hilbert space formalism defined by orthogonality of all pairs of rays where each ray is chosen in one of the subspaces) is now represented in the same way as orthogonality between the rays, i.e., by the orthogonality of two points of $\mathbb{R}^\nu$.

THEOREM 1b. *For all $a, b \in \mathscr{L}$, $a < b \Leftrightarrow r(a) \cdot r(b) = dim(a)$.*

*Proof.* Suppose that $a < b$. Let $\{p_1, \ldots, p_{dim(a)}\}$ correspond with a base of $a$ and that $\{p_1, \ldots, p_{dim(a)}, p_{dim(a)+1}, \ldots, p_{dim(b)}\}$ correspond with a base of $b$.

$$r(a) \cdot r(b) = \sum_{i=1}^{i=dim(a)} r(p_i) \cdot \sum_{i=1}^{i=dim(b)} r(p_i) = \sum_{i=1}^{i=dim(a)} r(p_i) \cdot r(p_i)$$

$$= \sum_{i=1}^{i=dim(a)} \left|\langle \psi_{p_i} | \psi_{p_i} \rangle\right|^2 = dim(a).$$

Let $\{\psi_{q_1}, \ldots, \psi_{q_{dim(b)}}\}$ be a base of $b$. For all $p \in \Sigma$, $r(p) \cdot \sum_{i=1}^{i=dim(b)} r(q_i) = \sum_{i=1}^{i=dim(b)} r(p) \cdot r(q_i) = \sum_{i=1}^{i=dim(b)} |\langle \psi_p | \psi_{q_i} \rangle|^2 \leq 1$, and we only can get an equality if $\psi_p \in b$, i.e., $p < b$. Thus we have $r(a) \cdot r(b) = \sum_{i=1}^{i=dim(a)} r(p_i) \cdot \sum_{i=1}^{i=dim(b)} r(q_i) \leq \dim(a)$, and we only can get an equality if $\forall i$, $\psi_{p_i} \in b$, i.e., $a < b$. ∎

This leads to the following corollary on the moduli of the points representative for the subspaces of $\mathscr{H}$:

COROLLARY 1. *For all $a \in \mathscr{L}$, $|r(a)| = \sqrt{dim(a)}$.*

Thus, the representation of the projectors on $k$-dimensional subspaces have length $\sqrt{k}$. If the Hilbert space is $n$-dimensional then $|r(1)| = \sqrt{n}$.

Since the representations of all equal dimension projectors have the same length, they all have the same angle with $r(1)$, the point in Euclidean space representative for the whole Hilbert space (the same reasoning can be made for all subspaces contained in any chosen subspace). If we denote the angle between the representations of two projectors $a$ and $b$ as $\theta(a,b)$ we can represent this observation in the following corollary:

COROLLARY 2. *For all* $a < b \in \mathscr{L}$, $\cos\theta(a,b) = \sqrt{dim(a)/dim(b)}$.

*Proof.* Since

$$\begin{aligned} dim(a) &= r(a)\cdot r(b) = |r(a)|\,|r(b)|\cos\theta(a,b) \\ &= \sqrt{dim(a)}\sqrt{dim(b)}\cos\theta(a,b) \end{aligned}$$

we have $\sqrt{dim(a)} = \sqrt{dim(b)}\cos\theta(a,b)$. ∎

If we express this result for the atoms in $\Sigma$ and for 1 we obtain the following remarkable result (this same remarkable property is valid for the subsets of $r(\mathscr{L})$ which correspond with subspaces of $\mathscr{H}$):

COROLLARY 3. *For all* $p \in \Sigma$, $\cos\theta(p,1) = \sqrt{1/n}$.

The last theorem on the logico-algebraical structure refers to the representation of the join.

THEOREM 1c. *For all* $a, b \in \mathscr{L}$ *with* $a \perp b$, $r(a \vee b) = r(a) + r(b)$.

*Proof.* If $\{p_1, \ldots, p_{dim(a)}\}$ corresponds with a base of $a$ and $\{q_1, \ldots, q_{dim(b)}\}$ corresponds with a base of $b$, then $\{p_1, \ldots, p_{dim(a)}, q_1, \ldots, q_{dim(b)}\}$ corresponds with a base of $a \vee b$. Thus, $r(a \vee b) = \sum_{i=1}^{i=dim(a)} r(p_i) + \sum_{i=1}^{i=dim(b)} r(q_i) = r(a) + r(b)$. ∎

The following theorem expresses the convex aspect of $\Omega$ within the representation space.

THEOREM 2. $r(\Omega)$ *is the convex closure of* $r(\Sigma)$.

*Proof.* The set $\mathscr{M}$ of all density matrices that corresponds with $\Omega$ is a convex set. Consider $f: \mathscr{M} \to \Omega$ that maps every density matrix onto the corresponding $\omega \in \Omega$. If one compares the explicit form of the density matrices with the representation $r$ (see Eq. (1), Eq. (2), ..., Eq. (10)), one easily sees that the convex structure of $\mathscr{M}$ is preserved by $r \circ f: \mathscr{M} \to r(\Omega)$ since, for a given density matrix $M$, the coordinates of $r(f(M))$ are in fact the real and the imaginary parts of the elements in the matrix, up to a constant (we omit an explicit proof since this requires rather heavy notations). ∎

In fact, we have more: the representation preserves convex combinations (this follows from the proof of the previous theorem).

COROLLARY 4. *For all* $\omega, \omega' \in \Omega$, $\alpha\omega + (1-\alpha)\omega' = \omega'' \Leftrightarrow \alpha r(\omega) + (1-\alpha)r(\omega') = r(\omega'')$.

For the representation introduced in the previous section, we have the following theorem on the representation of the convex set $\omega$ in relation to the representation of $\mathscr{H}$.

THEOREM 3. *For all* $a \in \mathscr{L}$, $\forall \omega \in \Omega$, $\omega(a) = r(\omega) \cdot r(a)$.

*Proof.* As a consequence of Lemma 2, $\forall p \in \Sigma$,

$$\omega(p) = \sum_{i=1}^{i=n} \omega(p_i^\omega) |\langle \psi_p | \psi_{p_i^\omega} \rangle|^2 = \sum_{i=1}^{i=n} \omega(p_i^\omega) r(p) \cdot r(p_i^\omega)$$
$$= r(p) \cdot \sum_{i=1}^{i=n} \omega(p_i^\omega) r(p_i^\omega) = r(\omega) \cdot r(p).$$

Let $\{p_1, \dots, p_{dim(a)}\}$ represent an orthonormal base of $a$. Then,

$$\omega(a) = \omega\big(p_1 \vee \cdots \vee p_{dim(a)}\big) = \sum_{i=1}^{i=dim(a)} \omega(p_i)$$
$$= \sum_{i=1}^{i=dim(a)} r(\omega) \cdot r(p_i) = r(\omega) \cdot r(a),$$

as a consequence of the additivity of $\omega$ on pairwise orthogonal states. ∎

## 4. CONCLUSION

We have constructed a map $r$ that enables a representation of the logico-algebraic and the convex approach together in the same representation space, in the sense that all the elements of the lattice $\mathscr{L}$ of closed subspaces of the Hilbert space and of the convex set $\Omega$ of normalized measures on $\mathscr{L}$ which are additive on mutual orthogonal elements are represented by a point in an Euclidean space, such that $r$ preserves some basic properties of $\mathscr{L}$ and $\Omega$. In some further investigation one could consider the following question: If one weakens some of the axioms of the lattice structure, could it still be possible to find such a joint representation which preserves some properties similar to those of Section 3?

## ACKNOWLEDGMENTS

We thank Professor J. Pykacz for his helpful remarks on this paper. We thank the referee for his concise remarks. The author is a Postdoctoral Researcher of the Flanders' Fund for Scientific Research. This work was supported by the ''Federale Diensten voor Wetenschappelijke, Technische en Culturele Aangelegenheden'' in the framework of IUAP-III, No. 9.

## REFERENCES

1. G. Birkhoff and J. von Neumann, *Ann. of Math.* **37** (1936), 823.
2. E. G. Beltrametti and G. Cassinelli, ''The Logic of Quantum Mechanics,'' Addison–Wesley, Reading, MA, 1981.
3. C. Piron, ''Foundations of Quantum Physics,'' Benjamin, London, 1976.
4. P. Pták and S. Pulmannova, ''Orthomodular Structures as Quantum Logics,'' Kluwer Academic, London, 1991.
5. E. I. Segal, *Ann. of Math.* **48** (1947), 930.
6. G. Ludwig, *Comm. Math. Phys.* **4** (1967), 3314.
7. K. E. Hellwig and K. Kraus, *Comm. Math. Phys.* **11** (1969), 214.
8. B. Mielnik, *Comm. Math. Phys.* **15** (1969), 1.
9. S. Gudder, *Comm. Math. Phys.* **29** (1973), 249.
10. D. Finkelstein, J. M. Jauch, and D. Speiser, *Helv. Phys. Acta* **32** (1959), 258.
11. D. Finkelstein, J. M. Jauch, S. Schimonovich, and D. Speiser, *J. Math. Phys.* **3** (1962), 207.
12. E. C. G. Stueckelberg, *Helv. Phys. Acta* **33** (1960), 727.
13. D. Aerts, *J. Math. Phys.* **27** (1986), 203.
14. B. Coecke, *Found. Phys.* **25** (1995), 1185.
15. B. Coecke, *Internat. J. Theoret. Phys.* **34** (1995), 1313.
16. C. Piron, *Helv. Phys. Acta* **37** (1964), 439.
17. V. S. Varadarajan, ''Geometry of Quantum Theory,'' Van Nostrand, Princeton, NH, 1968.
18. A. M. Gleason, *J. Math. Mech.* **6** (1957), 885.